

INHOMOGENEOUS JACOBI MATRICES ON TREES

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ABSTRACT. We study Jacobi matrices on trees with one end at infinity. We show that the defect indices cannot be greater than 1 and give criteria for essential selfadjointness. We construct certain polynomials associated with matrices which mimic orthogonal polynomials in the classical case. Nonnegativity of Jacobi matrices is studied as well.

1. INTRODUCTION

The aim of the paper is to study a special class of symmetric unbounded operators and their spectral properties. These are Jacobi operators defined on one sided tree. They are immediate generalizations of classical Jacobi matrices which act on sequences $\{u(n)\}_{n=0}^{\infty}$ by the rule

$$Ju(n) = \lambda_n u(n+1) + \beta_n u(n) + \lambda_{n-1} u(n-1), \quad n \geq 0,$$

where $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive and real numbers, respectively, with the convention $u(-1) = \lambda_{-1} = 0$. These matrices are closely related with the set of polynomials defined recursively by

$$(1) \quad xp_n(x) = \lambda_n p_{n+1}(x) + \beta_n p_n(x) + \lambda_{n-1} p_{n-1}(x), \quad n \geq 0,$$

with $p_{-1} = 0$, $p_0 = 1$.

In case the coefficients of the matrix are bounded, the matrix J represents selfadjoint operator on $\ell^2(\mathbb{N}_0)$. If $E(x)$ denotes the resolution of identity associated to J , then the polynomials $p_n(x)$ are orthonormal with respect to the measure $d\mu(x) = d(E(x)\delta_0, \delta_0)$, where δ_0 is the sequence taking value 1 at $n = 0$ and vanishing elsewhere and (u, v) denotes the standard inner product in $\ell^2(\mathbb{N}_0)$. The measure μ has bounded support.

When the coefficients are unbounded the operator J is well defined on the domain $D(J)$ consisting of sequences with finitely many nonzero terms. In that case, if this operator is essentially selfadjoint then again the polynomials p_n are orthonormal with respect to the measure $d\mu(x) = d(E(x)\delta_0, \delta_0)$, except that this measure has unbounded

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support. Moreover there is a unique orthogonality measure for polynomials p_n . By a classical theorem, if the operator J is not essentially selfadjoint, there are many measures μ on the real line so that the polynomials belong to $L^2(\mu)$, i.e.

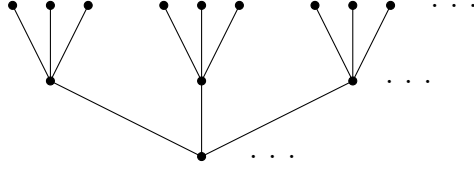
$$\int_{-\infty}^{\infty} x^{2n} d\mu(x) < \infty$$

and the polynomials p_n are orthogonal with respect to the inner product

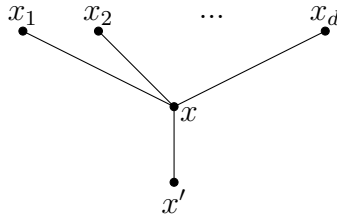
$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} d\mu(x).$$

Therefore essential selfadjointness is a crucial property that distinguishes between the so called determinate and indeterminate cases. Intuitively the unbounded matrix J is essentially selfadjoint when the coefficients have moderate growth. But the converse is not true in general. For the classical theory of Jacobi matrices, orthogonal polynomials and moment problems we address the reader to [1], [2], [5], and to [4] for a modern treatment.

In a recent paper [3] homogeneous Jacobi matrices on one sided homogeneous trees were studied. Two types of homogeneous trees were considered. One of them was the tree with infinitely many origin points (on level 0) and one end at infinity.



The tree Γ consists of vertices on levels from zero to infinity. Every vertex x on level n is connected with a unique vertex x' on level $n+1$ and d vertices x_1, \dots, x_d on level $n-1$ for $n \geq 1$ like in the figure below:



The Jacobi matrices were defined on $\ell^2(\Gamma)$ where Γ denotes the set of all vertices of the tree. The formula is as follows (when the degree

equals $d + 1$)

$$Jv(x) = \lambda_n v(x') + \beta_n v(x) + \lambda_{n-1} [v(x_1) + v(x_2) + \dots + v(x_d)],$$

where n denotes the level of the vertex x counting from above.

An interesting phenomenon occurred. It turned out that the operator J defined on functions $\{v(x)\}_{x \in \Gamma}$, with finitely many nonzero terms, is always essentially selfadjoint, regardless of the growth of the coefficients λ_n and β_n . For example the operator J with coefficients $\lambda_n = (n + 1)^2$ and $\beta_n = 0$ is not essentially selfadjoint when considered as the classical Jacobi matrix on $\ell^2(\mathbb{N}_0)$. But it is essentially selfadjoint when it acts on $\ell^2(\Gamma)$.

Moreover its spectrum is discrete and consists of the zeros of all the polynomials p_n associated with classical Jacobi matrix with coefficients $\sqrt{d} \lambda_n$ and β_n , i.e. satisfying

$$xp_n(x) = \sqrt{d} \lambda_n p_{n+1}(x) + \beta_n p_n(x) + \sqrt{d} \lambda_{n-1} p_{n-1}(x), \quad n \geq 0.$$

Our aim is to study inhomogeneous Jacobi matrix on that tree. This means we do not require that the coefficients of the matrix are constant on each level of the tree. With every vertex x we associate a positive number λ_x and a real number β_x . We are going to study operators of the form

$$Jv(x) = \lambda_x v(x') + \beta_x v(x) + \lambda_{x_1} v(x_1) + \lambda_{x_2} v(x_2) + \dots + \lambda_{x_d} v(x_d).$$

One of the main differences between the classical case and the case of the tree Γ is that the eigenvalue equation

$$(2) \quad zv(x) = \lambda_x v(x') + \beta_x v(x) + \lambda_{x_1} v(x_1) + \lambda_{x_2} v(x_2) + \dots + \lambda_{x_d} v(x_d)$$

cannot be solved recursively, unlike the equation

$$zv(n) = \lambda_n v(n + 1) + \beta_n v(n) + \lambda_{n-1} v(n - 1).$$

This not a coincidence as we are going to show that the equation (2) may not admit nonzero solutions for real values of z (cf. Proposition 5). But we will show the equation has a nonzero solution for every nonreal z (Corollary 3).

Actually, when we give up homogeneity of the matrix J , we can as well give up homogeneity of the tree. Again x' is the only vertex one level up connected with x and x_1, x_2, \dots, x_d are all vertices one level down connected with x . The number d may now vary if we do not assume homogeneity of the tree.

The operator J is symmetric on $\ell^2(\Gamma)$ with respect to the natural inner product

$$(u, v) = \sum_{x \in \Gamma} u(x) \overline{v(x)}.$$

We are interested in studying the essential selfadjointness of the matrix J . It turns out that unlike in homogeneous case, the matrix J may not be essentially selfadjoint. However the defect indices cannot be greater than 1 (Corollary 3). We derive certain criteria assuring essential selfadjointness. For example the analog of Carleman condition holds (see Theorem 6). Moreover we relate essential selfadjointness of J with essential selfadjointness of the classical Jacobi matrix J_0 obtained from J by restriction to an infinite path of the tree (see Theorem 4 and Remark following it).

Classical Jacobi matrices are associated with orthogonal polynomials through the formula (1). In case of the tree Γ there is no natural way of defining polynomials associated with Jacobi matrices on Γ , since (as was mentioned above) the eigenvalue equation may be not solvable. In Section 3 we define certain polynomials associated with J . We prove that they have real and simple zeros. Also we show interlacing property for roots of two consecutive polynomials. We also prove that the zeros of these polynomials describe the spectrum of restriction of J to finite subtrees of Γ . However, unlike in the classical case, there is no natural orthogonality relation between these polynomials.

In Section 5 we give a criterion for nonnegativity of the Jacobi matrix J on Γ . In the classical case the Jacobi matrix J is positive definite if and only if $(-1)^n p_n(0) > 0$ for every n , where p_n are the orthogonal polynomials associated with J . In case of tree Γ we do not have solutions of eigenvalue problem at our disposal or orthogonal polynomials. Therefore we had to find another way of getting the result. The nonnegativity of the matrix J proved to be a useful tool in construction of a Jacobi matrix on Γ for which the eigenvalue equation (2) does not admit solutions for some real values.

2. DEFINITIONS AND BASIC PROPERTIES

We will consider a tree Γ with one end at infinity. Its vertices are located on levels from zero to infinity. Every vertex x on level $k \geq 0$ is connected with a unique vertex x' on level $k + 1$. Moreover, when $k \geq 1$ the vertex x is connected with a finite number of vertices y on level $k - 1$. This set will be denoted by N_x . Let $\ell(x)$ denote the level of vertex x .

For a given vertex x let Γ_x denote the finite subtree containing the vertex x together with vertices y so that $\ell(y) < \ell(x)$ and connected with x by a path. Denote $\Gamma'_x = \Gamma_x \cup \{x'\}$.

Define $\mathcal{F}(\Gamma)$ to be the set of all complex valued functions with finite support on Γ . Let

$$\delta_x(y) = \begin{cases} 1 & y = x, \\ 0 & y \neq x. \end{cases}$$

Consider the operator J acting on $\mathcal{F}(\Gamma)$ according to the rule

$$(3) \quad J\delta_x = \lambda_x\delta_{x'} + \beta_x\delta_x + \sum_{y \in N_x} \lambda_y\delta_y, \quad \ell(x) > 1,$$

$$(4) \quad J\delta_x = \lambda_x\delta_{x'} + \beta_x\delta_x, \quad \ell(x) = 0,$$

where λ_x are positive constants while β_x are real ones. Let S be the operator acting by the rule

$$S\delta_x = \lambda_x\delta_{x'}.$$

Then the adjoint operator S^* is given by

$$S^*\delta_x = \begin{cases} \sum_{y \in N_x} \lambda_y\delta_y, & \ell(x) > 0, \\ S^*\delta_x = 0, & \ell(x) = 0. \end{cases}$$

Let M be a multiplication operator defined by

$$M\delta_x = \beta_x\delta_x.$$

Then

$$(5) \quad J = S + S^* + M.$$

In particular J is a symmetric linear operator.

We will study formal eignefunctions of the operator J , i.e. functions v defined on Γ satisfying

$$Jv = zv.$$

Evaluation at the vertex x gives that equivalently we have the recurrence relation

$$(6) \quad zv(x) = \begin{cases} \lambda_x v(x') + \beta_x v(x) + \sum_{y \in N_x} \lambda_y v(y) & \ell(x) \geq 1, \\ \lambda_x v(x') + \beta_x v(x) & \ell(x) = 0. \end{cases}$$

In order to simplify the notation set $N_x = \emptyset$ if $\ell(x) = 0$. Then (6) takes the form

$$(7) \quad zv(x) = \lambda_x v(x') + \beta_x v(x) + \sum_{y \in N_x} \lambda_y v(y)$$

Lemma 1. *Fix a vertex $x \in \Gamma$. Assume there exists $0 \neq v \in \mathcal{F}(\Gamma_{x'})$ and $z \notin \mathbb{R}$ such that $Jv(y) = zv(y)$ for $y \in \Gamma_x$. Then $v(x') \neq 0$.*

Proof. Assume for a contradiction that $v(x') = 0$. Let J_x denote the finite Jacobi matrix defined on Γ_x obtained from J by truncation. Namely $J_x = P_x J P_x$, where P_x denotes the orthogonal projection from $\ell^2(\Gamma)$ to $\ell^2(\Gamma_x)$. Let w denote the truncation of v to Γ_x . Since $v(x') = 0$ we have $J_x w = z w$. Moreover $w \neq 0$. Therefore z must be a real number, as J_x is a finite dimensional symmetric linear operator. \square

Lemma 2. *Fix a vertex $x \in \Gamma$. Assume there exists $0 \neq v \in \mathcal{F}(\Gamma_x)$ and $z \notin \mathbb{R}$ such that $Jv(y) = zv(y)$ for $y \in \Gamma_x \setminus \{x\}$. Then*

$$zv(x) \neq \beta_x v(x) + \sum_{y \in N_x} \lambda_y v(y).$$

Proof. Assume for a contradiction that

$$zv(x) = \beta_x v(x) + \sum_{y \in N_x} \lambda_y v(y).$$

Define the function $u \in \mathcal{F}(\Gamma_x)$ by setting $u(y) = v(y)$ for $y \in \Gamma_x$ and $u(x') = 0$. Then $Ju(y) = zu(y)$ for $y \in \Gamma_x$. In view of Lemma 1 we get a contradiction. \square

Corollary 1. *Assume there exists a function $v \neq 0$ on Γ and $z \notin \mathbb{R}$ such that $Jv(x) = zv(x)$ for $x \in \Gamma$. Then v does not vanish on Γ .*

Proof. Assume for a contradiction that $v(x) = 0$ for a vertex x . By Lemma 1 we get that the function v vanishes identically on Γ_x . From the recurrence relation

$$zv(x) = \lambda_x v(x') + \beta_x v(x) + \sum_{y \in N_x} \lambda_y v(y),$$

we get $v(x') = 0$. Therefore v vanishes identically on $\Gamma_{x'}$. Applying the same procedure infinitely many times we achieve that v vanishes at every vertex of Γ . \square

Lemma 3. *For any nonreal number z and any $x_0 \in \Gamma$ with $l(x_0) \geq 1$ there exists a nonzero function v defined on Γ_{x_0} satisfying*

$$(8) \quad Jv(x) = zv(x), \quad x \in \Gamma_{x_0} \setminus \{x_0\}.$$

Moreover the function v cannot vanish and is unique up to a constant multiple.

Proof. We will prove Lemma 3 by induction on the level $l(x_0)$. Assume $l(x_0) = 1$. Set $v(x_0) = 1$. Let $x \in N_{x_0}$. Then $l(x) = 0$. We want to have

$$zv(x) = \lambda_x v(x_0) + \beta_x v(x).$$

Thus we may set

$$v(x) = \frac{\lambda_x v(x_0)}{z - \beta_x}.$$

In this way (8) is fulfilled.

Assume the conclusion is true for all vertices on the level n . Let $l(x_0) = n + 1$. Consider vertices $x_1, x_2, \dots, x_k \in N_{x_0}$. Then $l(x_i) = n$ for $i = 1, 2, \dots, k$. By induction hypothesis, for every vertex x_i there exists a nonzero function v_i defined on Γ_{x_i} satisfying

$$Jv_i(x) = zv_i(x), \quad x \in \Gamma_{x_i} \setminus \{x_i\}.$$

We have

$$\Gamma_{x_0} = \bigcup_{i=1}^k \Gamma_{x_i} \cup \{x_0\}.$$

We are going to define the function v on Γ_{x_0} in the following way: set

$$v(x) = c_i v_i(x), \quad \text{for } x \in \Gamma_{x_i},$$

with $c_1 = 1$. In this way we get

$$Jv(x) = zv(x), \quad x \in \Gamma_{x_i} \setminus \{x_i\}, \quad i = 1, 2, \dots, k.$$

In order to conclude the proof we must show that

$$Jv(x_i) = zv(x_i), \quad i = 1, 2, \dots, k.$$

Thus we want to have

$$zc_i v_i(x_i) = \lambda_{x_i} v(x_0) + \beta_{x_i} c_i v_i(x_i) + \sum_{y \in N_{x_i}} \lambda_y c_i v_i(y),$$

i.e.

$$(9) \quad \lambda_{x_i} v(x_0) = c_i \left(zv_i(x_i) - \beta_{x_i} v_i(x_i) - \sum_{y \in N_{x_i}} \lambda_y v_i(y) \right).$$

The expression in the brackets on the right hand side is nonzero for every $i = 1, 2, \dots, k$ by Lemma 2. Therefore (9) is satisfied for an appropriate choice of the value $v(x_0)$ and nonzero constants c_2, c_3, \dots, c_k .

By Lemma 1 the function v cannot vanish. Moreover if there was another function \tilde{v} satisfying the conclusion of Lemma 3, then $v - c\tilde{v}$ would also satisfy the conclusion and would vanish for an appropriate choice of the constant c . Thus $v = c\tilde{v}$. \square

By the proof of Lemma 3 we get the following.

Corollary 2. *For any nonreal number z there exists a nonzero function v so that*

$$Jv(x) = zv(x), \quad x \in \Gamma.$$

The function v cannot vanish and is unique up to a constant multiple.

Remark. In Chapter 5 we are going to show that the conclusion of Corollary 2 may not be true for real numbers z . Observe that for classical Jacobi matrices (when $\Gamma = \mathbb{N}_0$) the recurrence relation

$$(10) \quad zv(n) = \lambda_n v(n+1) + \beta_n v(n) + \lambda_{n-1} v(n-1)$$

($\lambda_{-1} = 0$) has nonzero solutions for any $z \in \mathbb{C}$.

3. POLYNOMIALS AND ZEROS

The classical Jacobi matrices are related to orthogonal polynomials. Namely setting $v_0 = 1$ in (10) gives that $v(n) = p_n(z)$, where p_n is a polynomial of order n , with real coefficients. The question arises if Jacobi matrices on trees are connected to polynomials, as well. In general we cannot expect that the solution of $Jv = zv$ will satisfy that $v(t) = P_t(z)$, where P_t is a polynomial for every $t \in \Gamma$. But we may expect that $P_t(z)$ is a polynomial for t in a subtree Γ_x for some $x \in \Gamma$.

Proposition 1. *Let $x \in \Gamma$. There exists a nonzero solution v_x of $Jv_x = zv_x$, so that for any $t \in \Gamma'_x$ the function $v_x(t) = P_{x,t}(z)$ is a polynomial with real coefficients and positive leading coefficient. Moreover if $t \in \Gamma'_y \subset \Gamma'_x$ then the polynomial $P_{x,t}$ is divisible by $P_{y,t}$.*

Proof. We will use induction on the level $l(x)$. Let $l(x) = 0$. By Corollary 2 there is a nonzero solution v of $Jv = zv$. Then $v(x) \neq 0$. Let

$$v_x = v(x)^{-1}v.$$

Hence $v_x(x) = 1$. By $Jv_x = zv_x$ evaluated at x we get

$$v_x(x') = \frac{z - \beta_x}{\lambda_x}.$$

Assume now the conclusion is valid for vertices on the level n . Let $l(x) = n+1$. By induction hypothesis, for any $y \in N_x$ there is a nonzero solution v_y so that $P_{y,t}(z)$ is a polynomial with real coefficients for $t \in \Gamma'_y$. In particular the polynomial $v_y(x) = P_{y,x}(z)$ has real coefficients. Moreover by Lemma 1 the polynomial $P_{y,x}(z)$ cannot vanish for $z \notin \mathbb{R}$. Fix $y_1 \in N_x$ and let

$$(11) \quad v_x = \frac{\text{LCM}\{P_{y,x}(z) : y \in N_x\}}{P_{y_1,x}(z)} v_{y_1}.$$

Since $Jv_{y_1} = zv_{y_1}$ we get $Jv_x = zv_x$. Moreover v_x does not vanish for $z \notin \mathbb{R}$. Since $v_{y_1}(x) = P_{y_1,x}(z)$ we obtain

$$(12) \quad P_{x,x}(z) = v_x(x) = \text{LCM}\{P_{y',x}(z) : y' \in N_x\}.$$

Since the value $v_x(x)$ determines the solution, the function v_x does not depend on the choice of $y_1 \in N_x$. Thus the formula (11) and the above reasoning is valid for any choice of $y \in N_x$. Hence

$$(13) \quad v_x = \frac{\text{LCM}\{P_{\tilde{y},x}(z) : \tilde{y} \in N_x\}}{P_{y,x}(z)} v_y, \quad y \in N_x.$$

By (13) and by induction hypothesis the value $v_x(t)$ is a polynomial in z for any $t \in \bigcup_{y \in N_x} \Gamma_y \cup \{x\} = \Gamma_x$. By the recurrence relation also the value $v_x(x')$ is a polynomial. Moreover by (13) the polynomial $v_x(t)$ is divisible by $v_y(t)$ for any $y \in N_x$ and $t \in \Gamma'_y$. This implies the last part of the conclusion. \square

Remarks The formulas (12) and (13) imply that for $y \in N_x$ and $t \in \Gamma'_y$ we have

$$(14) \quad P_{x,t}(z) = P_{x,x}(z) \frac{P_{y,t}(z)}{P_{y,x}(z)}.$$

Let $y \in \Gamma_x$. Then y and x are connected in Γ_x by a path $y = y_0, y_1, \dots, y_n = x$. By iterating (14) we get

$$P_{x,y}(z) = \frac{P_{y_n,y_n}(z)}{P_{y_{n-1},y_n}(z)} \cdot \frac{P_{y_{n-1},y_{n-1}}(z)}{P_{y_{n-2},y_{n-1}}(z)} \cdot \dots \cdot \frac{P_{y_1,y_1}(z)}{P_{y_0,y_1}(z)} P_{y_0,y_0}(z).$$

Let $y \in \Gamma_{\tilde{x}} \subset \Gamma_x$. Then $\tilde{x} = y_k$ for some k , $0 \leq k \leq n$. Hence

$$P_{x,y}(z) = \frac{P_{y_n,y_n}(z)}{P_{y_{n-1},y_n}(z)} \cdot \frac{P_{y_{n-1},y_{n-1}}(z)}{P_{y_{n-2},y_{n-1}}(z)} \cdot \dots \cdot \frac{P_{y_{k+1},y_{k+1}}(z)}{P_{y_k,y_{k+1}}(z)} P_{\tilde{x},y}(z).$$

These formulas and (12) imply that the polynomial $P_{x,y}(z)$ can be described in terms of the polynomials of the form $P_{t,t'}(z)$ for $t \in \Gamma_x$.

Corollary 3. *Let $z \notin \mathbb{R}$. Let $\{x_n\}_{n=0}^\infty$ be an infinite path in Γ so that $l(x_n) = n$. Let v be a nonzero solution of $Jv(x) = zv(x)$ so that $v(x_0) = 1$. Then for any vertex $x \in \Gamma_{x_n}$ we have*

$$v(x) = \frac{a_{n,x}(z)}{b_n(z)},$$

where $a_{n,x}(z)$ and $b_n(z)$ are polynomials with real coefficients. Moreover the polynomial b_{n+1} is divisible by b_n .

Proof. Consider the subtree Γ_{x_n} . Let $x \in \Gamma_{x_n}$. By Proposition 1 there is a solution v_n so that $v_n(x)$ and $v_n(x_0)$ are polynomials with real coefficients. Then

$$v(x) = \frac{v_n(x)}{v_n(x_0)}$$

satisfies $v(x_0) = 1$. By the last part of Proposition 1 the polynomial $v_{n+1}(x_0)$ is divisible by $v_n(x_0)$. \square

Theorem 1. *The polynomials $P_{x,y}(z)$, $y \in \Gamma'_x$, have only real zeros. Moreover for any $x \in \Gamma$ the zeros of $P_{x,x}$ and $P_{x,x'}$ are single, and the zeros of $P_{x,x}$ interlace with the zeros of $P_{x,x'}$, i.e. if $x_1 < x_2 < \dots < x_n$ denote the zeros of $P_{x,x'}$, then $P_{x,x}$ has $n-1$ zeros $y_1 < y_2 < \dots < y_{n-1}$ and*

$$x_1 < y_1 < x_2 < y_2 < \dots < y_{n-1} < x_n.$$

Proof. We will use induction on $l(x)$. Let $l(x) = 0$. Then $P_{x,x} = 1$ and $P_{x,x'} = (z - \beta_x)/\lambda_x$. Assume the conclusion is valid for $l(x) = n-1$. Let $l(x) = n$. By the recurrence relation we have

$$(15) \quad \lambda_x P_{x,x'}(z) = (z - \beta_x) P_{x,x}(z) - \sum_{j=1}^k \lambda_{y_j} P_{x,y_j}(z),$$

where $N_x = \{y_1, y_2, \dots, y_k\}$. By (14), with $t = y_j$, we get

$$(16) \quad P_{x,y_j}(z) = P_{x,x}(z) \frac{P_{y_j,y_j}(z)}{P_{y_j,x}(z)}.$$

By induction hypothesis the zeros of $P_{y_j,y_j}(z)$ are real and single and interlace with the zeros of $P_{y_j,x}(z)$ for any j . This implies

$$\deg P_{y_j,x} = \deg P_{y_j,y_j} + 1.$$

In view of (15) and (16) we get

$$\deg P_{x,x'} = \deg P_{x,x} + 1.$$

Let r be a root of $P_{x,x}(z)$. We are going to study the sign of $P_{x,x'}(r)$ making use of (15). If $P_{y_j,x}(r) \neq 0$, then (16) implies $P_{x,y_j}(r) = 0$. But since $P_{x,x}(r) = 0$ then $P_{y_{j_0},x}(r_1) = 0$ for some j_0 , by (12). Consider the quantity

$$P_{x,y_{j_0}}(r + \varepsilon) = P_{x,x}(r + \varepsilon) \frac{P_{y_{j_0},y_{j_0}}(r + \varepsilon)}{P_{y_{j_0},x}(r + \varepsilon)},$$

where $\varepsilon > 0$ is infinitesimally small. We have

$$\frac{P_{y_{j_0},y_{j_0}}(r + \varepsilon)}{P_{y_{j_0},x}(r + \varepsilon)} > 0,$$

as the polynomials $P_{y_{j_0}, y_{j_0}}(z)$ and $P_{y_{j_0}, x}(z)$ have the same number of roots to the right of $r + \varepsilon$, by induction hypothesis and by the fact that the leading coefficients are positive. Consider the limit

$$\begin{aligned}
 P_{x, y_{j_0}}(r) &= \lim_{\varepsilon \rightarrow 0^+} P_{x, x}(r + \varepsilon) \frac{P_{y_{j_0}, y_{j_0}}(r + \varepsilon)}{P_{y_{j_0}, x}(r + \varepsilon)} \\
 (17) \qquad &= P_{y_{j_0}, y_{j_0}}(r) \lim_{\varepsilon \rightarrow 0^+} \frac{P_{x, x}(r + \varepsilon)}{P_{y_{j_0}, x}(r + \varepsilon)}.
 \end{aligned}$$

The polynomials $P_{y, x}$, for $y \in N_x$, have single roots by induction hypothesis. Thus the limit in the right hand side of (17) is nonzero in view of (12). Since $P_{y_{j_0}, y_{j_0}}(r) \neq 0$ (by induction hypothesis) we get that $P_{x, j_0}(r) \neq 0$. Hence the sign of the limit is determined by the sign of $P_{x, x}(r + \varepsilon)$. By plugging $z = r$ into (15) we get that $P_{x, x'}(r)$ and $P_{x, x}(r + \varepsilon)$ have opposite signs.

Consider now two consecutive roots $r_1 < r_2$ of $P_{x, x}(z)$. The signs of $P_{x, x}(r_1 + \varepsilon)$ and $P_{x, x}(r_2 + \varepsilon)$ are opposite. Therefore the signs of $P_{x, x'}(r_1)$ and $P_{x, x'}(r_2)$ are also opposite. Thus $P_{x, x'}(z)$ must vanish in the interval (r_1, r_2) .

Assume now that r is the largest root of $P_{x, x}(z)$. Then $P_{x, x}(r + \varepsilon) > 0$ for small positive ε . By the above reasoning we have $P_{x, x'}(r) < 0$, which means that $P_{x, x'}$ must vanish somewhere to the right of r , as the leading coefficient is positive. Similarly if r is the smallest root of $P_{x, x}(z)$ then the signs of $P_{x, x'}(r)$ and $P_{x, x}(r + \varepsilon)$ are opposite. But since the degree of $P_{x, x'}$ is by one greater than the degree of $P_{x, x}(z)$ and the leading coefficients are positive, we get that $P_{x, x'}$ must vanish below r . \square

For $x \in \Gamma$ let J_x denote the truncation of the Jacobi matrix J to the subtree Γ_x , i.e. the matrix with the parameters λ_y^x, β_y^x so that

$$\lambda_y^x = \begin{cases} \lambda_y & \text{for } y \in \Gamma_x \setminus \{x\} \\ 0 & \text{for } y \notin \Gamma_x \setminus \{x\} \end{cases} \qquad \beta_y^x = \begin{cases} \beta_y & \text{for } y \in \Gamma_x \\ 0 & \text{for } y \notin \Gamma_x \end{cases}$$

Theorem 2. *Let $x \in \Gamma$. Let r belong to the spectrum of J_x . Then r satisfies at least one of the two conditions*

- (a) $P_{x, x'}(r) = 0$.
- (b) *There exist $y \in \Gamma_x$ and $y_1, y_2 \in N_y$ so that $P_{y_1, y}(r) = P_{y_2, y}(r) = 0$.*

Proof. First we will show that the numbers described in the theorem belong to the spectrum of J_x . Assume $P_{x, x'}(r) = 0$. By Theorem 1 we have $P_{x, x}(r) \neq 0$. By Lemma 2 for any nonreal z there is a solution v_x

of the equation $Jv_x = zv_x$ so that $v_x(y) = P_{x,y}(z)$ for $y \in \Gamma'_x$. Let

$$u(y) = \lim_{\varepsilon \rightarrow 0} P_{x,y}(r + i\varepsilon), \quad y \in \Gamma_x.$$

Then u satisfies $J_x u = ru$. Moreover u is nonzero as $u(x) = P_{x,x}(r) \neq 0$.

Assume now that there exist $y \in \Gamma_x$ and $y_1, y_2 \in N_y$ so that $P_{y_1,y}(r) = P_{y_2,y}(r) = 0$. By the above reasoning there are two nonzero solutions u_1, u_2 , defined on $\Gamma_{y_1}, \Gamma_{y_2}$, respectively, of the equations $J_{y_1} u_1 = ru_1$ and $J_{y_2} u_2 = ru_2$ and $u_1(y_1) \neq 0, u_2(y_2) \neq 0$. Consider the function u_{y_1,y_2} defined on Γ_y as follows

$$u_{y_1,y_2}(t) = \begin{cases} \lambda_{y_2} u_2(y_2) u_1(t) & t \in \Gamma_{y_1}, \\ -\lambda_{y_1} u_1(y_1) u_2(t) & t \in \Gamma_{y_2}, \\ 0 & t \notin \Gamma_{y_1} \cup \Gamma_{y_2}. \end{cases}$$

Then $u_{y_1,y_2} \neq 0$ and $J_x u_{y_1,y_2} = J u_{y_1,y_2} = ru_{y_1,y_2}$.

Assume that for $y \in \Gamma_x$ and $y_1, y_2, \dots, y_n \in N_y$ we have

$$P_{y_1,y}(r) = P_{y_2,y}(r) = \dots = P_{y_n,y}(r) = 0.$$

Then the eigenvectors

$$u_{y_1,y_2}, \dots, u_{y_1,y_n}$$

are linearly independent, as the support of u_{y_1,y_i} coincides with $\Gamma_{y_1} \cup \Gamma_{y_i}$. Hence the dimension of the space spanned by these eigenvectors is at least $n - 1$.

In the previous part of the proof we have constructed eigenvectors corresponding to the set of numbers described in the theorem. We will calculate the dimension of the space spanned by these eigenvectors. The proof will be complete if the dimension coincides with the dimension of the space $\ell^2(\Gamma_x)$, i.e. with $\#\Gamma_x$. We will use induction with respect to $\ell(x)$. Assume the conclusion is valid for $\ell(x) = n$. Let $\ell(x) = n + 1$. Denote $N_x = \{y_1, y_2, \dots, y_k\}$. Let $n_j = \deg P_{y_j,x}$. Every eigenvector of J_{y_j} corresponding to the case (b) is an eigenvector of J_x as well. Therefore, by induction hypothesis, the dimension of the linear span of all eigenvectors of J_{y_i} corresponding to the case (b) is equal

$$\#\Gamma_{y_i} - \deg P_{y_i,x}.$$

Such eigenvectors corresponding to J_{y_i} and J_{y_j} , for $i \neq j$ have disjoint supports, hence the total dimension of the eigenvectors corresponding to the case (b) for J_{y_1}, \dots, J_{y_k} is equal

$$\sum_{j=1}^k \#\Gamma_{y_j} - \sum_{j=1}^k \deg P_{y_j,x}$$

Consider the product

$$P_{y_1,x}(z) \dots P_{y_k,x}(z).$$

We know that every polynomial $P_{y_j,x}$ has single roots. We have

$$P_{y_1,x}(z) \dots P_{y_k,x}(z) = c \prod_{l=1}^L (z - r_l)^{n_l}.$$

By the reasoning performed in the first part of the proof, the root r_l gives rise to $n_l - 1$ linearly independent eigenvectors of J_x . Moreover the degree of the polynomial $P_{x,x'}$ is equal to $L + 1$ as

$$\deg P_{x,x'} = \deg P_{x,x} + 1,$$

and $\deg P_{x,x} = L$ (cf. (12)). The roots of $P_{x,x'}$ lead to $L + 1$ linearly independent eigenvectors of J_x , which are linearly independent from the ones constructed in (b), as they do not vanish at x . Summarizing the number of linearly independent eigenvectors of J_x is not less than

$$\sum_{j=1}^k \# \Gamma_{y_j} - \sum_{j=1}^k \deg P_{y_j,x} + \sum_{l=1}^L (n_l - 1) + L + 1 = \sum_{j=1}^k \# \Gamma_{y_j} + 1 = \# \Gamma_x.$$

□

4. ESSENTIAL SELFADJOINTNESS AND DEFECT INDICES

Let $z \notin \mathbb{C}$. The function $v \in \ell^2(\Gamma)$ belongs to the defect space corresponding to \bar{z} if v is orthogonal to $\text{Im}(\bar{z}I - J) = (\bar{z}I - J)(\mathcal{F}(\Gamma))$. In particular v is orthogonal to $(\bar{z}I - J)\delta_x$ for any $x \in \Gamma$. This implies $Jv = zv$.

Proposition 2. *The defect indices of the operator J cannot be greater than 1.*

Proof. Fix a nonreal number z . Let $v \in \ell^2(\Gamma)$ satisfy $v \neq 0$ and $Jv = zv$. By Corollary 1 the function v is unique up to a constant multiple. □

Proposition 2 implies

Corollary 4. *Let J be a Jacobi matrix on Γ . Fix a nonreal number z and let v denote the unique, up to a constant multiple, nonzero solution of the equation $Jv = zv$. Then J is essentially selfadjoint if and only if $v \notin \ell^2(\Gamma)$.*

Theorem 3. *There exist essentially nonselfadjoint Jacobi matrices on Γ .*

Proof. We set $\beta_x \equiv 0$. Fix a nonreal number z . Choose an infinite path x_n in Γ so that $l(x_n) = n$. We will construct a matrix J by induction on n . Assume we have constructed a matrix J on $\Gamma_{x_{n-1}} \setminus \{x_{n-1}\}$ and a nonvanishing function v on $\Gamma_{x_{n-1}}$ so that

$$\|v|_{\Gamma_{x_{n-1}}}\|_2^2 \leq 1 - 2^{-(n-1)}$$

and

$$Jv(x) = zv(x), \quad x \in \Gamma_{x_{n-1}} \setminus \{x_{n-1}\}.$$

We want to extend the definition of J and v so that the conclusion remains valid when $n-1$ is replaced by n .

Our first task is to define $\lambda_{x_{n-1}}$ and $v(x_n)$ so that

$$zv(x_{n-1}) = \lambda_{x_{n-1}}v(x_n) + \sum_{y \in N_{x_{n-1}}} \lambda_y v(y),$$

i.e.

$$(18) \quad \lambda_{x_{n-1}}v(x_n) = zv(x_{n-1}) - \sum_{y \in N_{x_{n-1}}} \lambda_y v(y).$$

The right hand side of (18) cannot vanish by Lemma 2. We will define $\lambda_{x_{n-1}}$ and $v(x_n)$ so as to satisfy (18). By specifying $\lambda_{x_{n-1}}$ large enough we may assume that

$$|v(x_n)|^2 \leq 2^{-n-1}.$$

For any $y \in N_{x_n}$ and $y \neq x_{n-1}$ consider the subtree $\Gamma_y \setminus \{y\}$. Set $\lambda_x = 1$ for any $x \in \Gamma_y \setminus \{y\}$. By Lemma 3 there is a nonzero solution v_y defined on Γ_y satisfying

$$Jv_y(x) = zv_y(x), \quad x \in \Gamma_y \setminus \{y\}.$$

We may assume that

$$\sum_{y \in N_{x_n} \setminus \{x_{n-1}\}} \|v_y|_{\Gamma_y}\|_2^2 \leq 2^{-n-1}.$$

We want to define the numbers λ_y for $y \in N_{x_n}$ and $y \neq x_{n-1}$ so that

$$zv_y(y) = Jv(y) = \lambda_y v_y(x_n) + \sum_{x \in N_y} \lambda_x v_y(x).$$

Hence we want to have

$$(19) \quad \lambda_y = \frac{zv_y(y) - \sum_{x \in N_y} \lambda_x v_y(x)}{v(x_n)}.$$

By Lemma 2 the numerator (19) cannot vanish. We may multiply v_y by a constant of absolute value 1 so that the expression on the right hand side of (19) becomes positive. In this way the values λ_y for $y \in N_{x_n}$

and $y \neq x_{n-1}$ are defined. We extend the definition of v to Γ_{x_n} by setting

$$v(x) = v_y(x), \quad x \in \Gamma_y, \quad y \neq x_{n-1}.$$

On the way we have also extended the definition of J so that

$$Jv(x) = zv(x), \quad x \in \Gamma_{x_n} \setminus \{x_n\}.$$

Moreover by construction we have

$$\begin{aligned} \|v|_{\Gamma_{x_n}}\|_2^2 &= \|v|_{\Gamma_{x_{n-1}}}\|_2^2 + \sum_{y \in N_{x_n}, y \neq x_{n-1}} \|v|_{\Gamma_y}\|_2^2 + |v(x_n)|^2 \\ &\leq 1 - 2^{-(n-1)} + 2^{-n-1} + 2^{-n-1} = 1 - 2^{-n}. \end{aligned}$$

□

Remark 1. The Jacobi matrix J constructed in the proof satisfies $\beta_x \equiv 0$ and $\lambda_x = 1$ for vertices x whose distance from the path $\{x_n\}$ is greater than 2.

Remark 2. Another way of proving Theorem 3 is as follows. Fix any Jacobi matrix J_0 so that the operator J_0 is bounded on $\ell^2(\Gamma)$. For example Let $\beta_x \equiv 0$ and $\lambda_x = d_x^{-1/2}$, where $d_x = \#N_{x'}$. Let S denote the operator acting according to the rule

$$Sv(x) = \lambda_x v(x').$$

Then

$$\begin{aligned} \|Sv\|_2^2 &= \sum_{x \in \Gamma} |Sv(x)|^2 = \sum_{x \in \Gamma} \lambda_x^2 |v(x')|^2 \\ &= \sum_{x' \in \Gamma \setminus \Gamma_0} |v(x')|^2 \sum_{x \in N_{x'}} \lambda_x^2 = \sum_{x' \in \Gamma \setminus \Gamma_0} |v(x')|^2 \leq \|v\|_2^2. \end{aligned}$$

The operator S is thus bounded. The adjoint operator S^* acts by the rule

$$\begin{aligned} S^*v(x) &= \sum_{y \in N_x} \lambda_y v(y), & x \notin \Gamma_0, \\ S^*v(x) &= 0, & x \in \Gamma_0. \end{aligned}$$

Then $J_0 = S + S^*$ is the Jacobi matrix such that $\|J_0\|_{2 \rightarrow 2} \leq 2$. Fix an infinite path $\{x_n\}$ and a sequence of positive numbers $\{\lambda_n\}$. Let J_1 be the degenerate Jacobi matrix defined by $\beta_x \equiv 0$ and $\lambda_{x_n} = \lambda_n$, $\lambda_x = 0$ for $x \notin \{x_n\}$. Choose the coefficients λ_n so that the classical Jacobi matrix associated with the coefficients λ_n and $\beta_n \equiv 0$ is not essentially selfadjoint. For example let $\lambda_n = 2^{-n}$. Let $J = J_0 + J_1$. The matrix J is nondegenerate. Moreover J is not essentially selfadjoint as a bounded perturbation of not essentially selfadjoint operator.

The next theorem provides a relation between Jacobi matrices on the tree Γ and classical Jacobi matrices associated with the infinite paths of Γ .

The Jacobi matrix J on Γ will be called *symmetric* if $\beta_x \equiv 0$.

Theorem 4. *Let J be a nonessentially selfadjoint symmetric Jacobi matrix on Γ . Choose an infinite path $\{x_n\}$ so that $l(x_n) = n$. Then the classical Jacobi matrix J_0 with $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$ is nonessentially selfadjoint.*

Before proving the theorem we will need the following lemma.

Lemma 4. *Let J be a symmetric Jacobi matrix on Γ . Let $Jv = iv$ and $v(x_0) = 1$ for a vertex x_0 on the level 0. Then the function $\tilde{v}(x) = i^{-l(x)}v(x)$ is positive.*

Proof. By assumptions we have

$$iv(x) = \lambda_x v(x') + \sum_{y \in N_x} \lambda_y v(y).$$

Thus

$$(20) \quad \tilde{v}(x) = \lambda_x \tilde{v}(x') - \sum_{y \in N_x} \lambda_y \tilde{v}(y).$$

We know that \tilde{v} cannot vanish and is unique up to a constant multiple. The function $\operatorname{Re} \tilde{v}$ satisfies (20) and takes the value 1 at x_0 . Thus $\tilde{v} = \operatorname{Re} \tilde{v}$, i.e. \tilde{v} is real valued. We will show that $\tilde{v}(x)$ is positive by induction. Observe that if $\tilde{v}(x)$ is positive for any vertex on level zero then by (20) \tilde{v} is positive. Assume the opposite, i.e. \tilde{v} is negative at some vertices on level zero. Since $\tilde{v}(x_0) = 1$ there are two vertices y_1, y_2 on level zero so that $y'_1 = y'_2$ and $\tilde{v}(y_1) > 0$, $\tilde{v}(y_2) < 0$. By (20) evaluated at $x = y_1$ and $x = y_2$ we get that $\tilde{v}(y'_1) > 0$ and $\tilde{v}(y'_2) < 0$, which gives a contradiction. □

Proof of Theorem 3. By (20) evaluated at $x = x_n$ we obtain

$$\tilde{v}(x_n) = \lambda_{x_n} \tilde{v}(x_{n+1}) - \sum_{y \in N_{x_n}} \lambda_y \tilde{v}(y).$$

Hence

$$\lambda_{x_n} \tilde{v}(x_{n+1}) - \lambda_{x_{n-1}} \tilde{v}(x_{n-1}) = \tilde{v}(x_n) + \sum_{\substack{y \in N_{x_n} \\ y \neq x_{n-1}}} \lambda_y \tilde{v}(y) > 0.$$

The last inequality follows from Lemma 4. Therefore

$$\begin{aligned}\tilde{v}(x_{2n}) &\geq \frac{\lambda_{x_0}\lambda_{x_2}\cdots\lambda_{x_{2n-2}}}{\lambda_{x_1}\lambda_{x_3}\cdots\lambda_{x_{2n-1}}}\tilde{v}(x_0), \\ \tilde{v}(x_{2n+1}) &\geq \frac{\lambda_{x_1}\lambda_{x_3}\cdots\lambda_{x_{2n-1}}}{\lambda_{x_2}\lambda_{x_4}\cdots\lambda_{x_{2n}}}\tilde{v}(x_1).\end{aligned}$$

By assumptions the sequence $\tilde{v}(x_n)$ is square summable. Thus

$$(21) \quad \sum_{n=1}^{\infty} \left(\frac{\lambda_{x_0}\lambda_{x_2}\cdots\lambda_{x_{2n-2}}}{\lambda_{x_1}\lambda_{x_3}\cdots\lambda_{x_{2n-1}}} \right)^2 + \left(\frac{\lambda_{x_1}\lambda_{x_3}\cdots\lambda_{x_{2n-1}}}{\lambda_{x_2}\lambda_{x_4}\cdots\lambda_{x_{2n}}} \right)^2 < \infty.$$

The last inequality is equivalent to nonessential selfadjointness of the classical Jacobi matrix J_0 with $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$. Indeed, let p_n and q_n denote the polynomials of the first and the second kind associated with J_0 . Then (21) reduces to

$$\sum_{n=1}^{\infty} [p_n^2(0) + q_n^2(0)] < \infty.$$

□

Remark The symmetry assumption $\beta_x \equiv 0$ is essential. There exist nonessentially selfadjoint Jacobi matrices J on Γ so that the classical Jacobi matrix J_0 associated with the path $\{x_n\}$ is essentially selfadjoint. Indeed for every vertex x_n , $n \geq 1$, fix a vertex $y_{n-1} \neq x_{n-1}$ in N_{x_n} . Let P denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\{x_n, y_n\}_{n=0}^{\infty})$. We will consider Jacobi matrices J so that $\beta_x = 0$ for $x \notin \{y_n\}_{n=0}^{\infty}$. Let $J_1 = PJP$. First we are going to study the essential selfadjointness of the operator J_1 . To this end consider the equation $J_1 v(x) = zv(x)$. This is equivalent to

$$\begin{aligned}zv(x_n) &= \lambda_{x_n}v(x_{n+1}) + \lambda_{x_{n-1}}v(x_{n-1}) + \lambda_{y_{n-1}}v(y_{n-1}), \\ zv(y_{n-1}) &= \lambda_{y_{n-1}}v(x_n) + \beta_{y_{n-1}}v(y_{n-1}).\end{aligned}$$

We have

$$(22) \quad v(y_{n-1}) = \frac{\lambda_{y_{n-1}}}{z - \beta_{y_{n-1}}}v(x_n).$$

Hence

$$zv(x_n) = \lambda_{x_n}v(x_{n+1}) + \lambda_{x_{n-1}}v(x_{n-1}) + \frac{\lambda_{y_{n-1}}^2}{z - \beta_{y_{n-1}}}v(x_n).$$

Set $z = i$, $v_n := v_{x_n}$, $\mu_n := \lambda_{y_{n-1}}$, $\lambda_n := \lambda_{x_n}$ and $\beta_n := \beta_{y_{n-1}}$. Then

$$\left[\frac{\beta_n \mu_n^2}{1 + \beta_n^2} + \left(1 + \frac{\mu_n^2}{1 + \beta_n^2} \right) i \right] v_n = \lambda_n v_{n+1} + \lambda_{n-1} v_{n-1}.$$

Set $\mu_n^2 = 1 + \beta_n^2$. Then we obtain

$$2iv_n = \lambda_n v_{n+1} - \beta_n v_n + \lambda_{n-1} v_{n-1}.$$

Assume the classical Jacobi matrix with coefficients λ_n and $-\beta_n$ is not essentially selfadjoint. Then the sequence v_n is square summable. Moreover (22) implies

$$|v(y_{n-1})|^2 = \frac{\mu_n^2}{1 + \beta_n^2} |v_n|^2 = |v_n|^2.$$

Hence

$$\|v\|^2 = \sum_{n=0}^{\infty} |v(x_n)|^2 + |v(y_n)|^2 < \infty,$$

i.e. the operator J_1 is not essentially selfadjoint. Let J_2 be any bounded Jacobi matrix on Γ . Then $J = J_1 + J_2$ is a nonessentially selfadjoint Jacobi matrix on Γ .

The matrix J_0 is associated with the coefficients $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$. Thus, in order to conclude the reasoning it suffices to prove the following.

Lemma 5. *There exists a nonselfadjoint classical Jacobi matrix J*

$$Jx_n = \lambda_n x_{n+1} - \beta_n x_n + \lambda_{n-1} x_{n-1}$$

so that the Jacobi matrix

$$J'x_n = \lambda_n x_{n+1} + \lambda_{n-1} x_{n-1}$$

is essentially selfadjoint.

Proof. We will assume that $\beta_n \neq 0$. Nonessential selfadjointness of J is equivalent to the fact that every solution of the recurrence relation

$$0 = \lambda_n x_{n+1} - \beta_n x_n + \lambda_{n-1} x_{n-1}, \quad n \geq 1,$$

is square summable. Assume the sequence x_n satisfies this recurrence relation. Then

$$\begin{aligned} (23) \quad \beta_{2n} x_{2n} &= \lambda_{2n} x_{2n+1} + \lambda_{2n-1} x_{2n-1}, \\ \beta_{2n+1} x_{2n+1} &= \lambda_{2n+1} x_{2n+2} + \lambda_{2n} x_{2n}. \end{aligned}$$

Thus

$$\begin{aligned} (24) \quad x_{2n+1} &= \frac{\lambda_{2n+1}}{\beta_{2n+1}} x_{2n+2} + \frac{\lambda_{2n}}{\beta_{2n+1}} x_{2n}, \\ x_{2n-1} &= \frac{\lambda_{2n-1}}{\beta_{2n-1}} x_{2n} + \frac{\lambda_{2n-2}}{\beta_{2n-1}} x_{2n-2} \end{aligned}$$

Plugging in the last two equations into (23) results in

$$\left(\beta_{2n} - \frac{\lambda_{2n}^2}{\beta_{2n+1}} - \frac{\lambda_{2n-1}^2}{\beta_{2n-1}} \right) x_{2n} = \frac{\lambda_{2n}\lambda_{2n+1}}{\beta_{2n+1}} x_{2n+2} + \frac{\lambda_{2n-2}\lambda_{2n-1}}{\beta_{2n-1}} x_{2n-2}.$$

Let $\beta_{2n-1} = a\lambda_{2n-1}$ and

$$\beta_{2n} = \frac{\lambda_{2n}^2}{\beta_{2n+1}} + \frac{\lambda_{2n-1}^2}{\beta_{2n-1}}.$$

Then

$$0 = \lambda_{2n}x_{2n+2} + \lambda_{2n-2}x_{2n-2}.$$

Choose an increasing sequence λ_{2n} so that every solution u_{2n} of the last equation is square summable. Assume also that $\lambda_{2n} = \lambda_{2n+1}$. Then by (24) we get

$$|x_{2n+1}| \leq |a||x_{2n+2}| + \frac{\lambda_{2n}}{\lambda_{2n+1}}|x_{2n}| = |a||x_{2n+2}| + |x_{2n}|.$$

Thus the sequence x_n is square summable, i. e. the Jacobi matrix J is not essentially selfadjoint.

On the other hand the Jacobi matrix J' , under assumption $\lambda_{2n} = \lambda_{2n+1}$, is essentially selfadjoint. Indeed, the sequence $x_{2n-1} = 0$ and $x_{2n} = (-1)^n$ satisfies $J'x = 0$ and it is not square summable. \square

Remark. Following the proof it is possible to construct the coefficients λ_n and β_n explicitly. Let $\lambda_{2n+1} = \lambda_{2n} = q^n$ for $q > 1$. Then $\beta_{2n+1} = aq^n$ and $\beta_{2n} = a^{-1}[q^n + q^{n-1}]$.

The following lemma is straightforward but useful.

Lemma 6. Consider a symmetric operator A on a Hilbert space \mathcal{H} . Let \mathcal{H}_0 be a finite dimensional subspace of $D(A) \subset \mathcal{H}$ and let $P_{\mathcal{H}_0}$ denote the orthogonal projection onto \mathcal{H}_0 . Define the operator $\tilde{A} : \mathcal{H}_0^\perp \rightarrow \mathcal{H}_0^\perp$ by

$$\tilde{A} = (I - P_{\mathcal{H}_0})A(I - P_{\mathcal{H}_0}).$$

The operator \tilde{A} is essentially selfadjoint if and only if A is essentially selfadjoint.

Theorem 5. Assume J is not essentially selfadjoint. Fix a vertex x_0 on the level zero and a nonreal number z . Let a function $u(x)$ satisfy $u \neq 0$, $u(x_0) = 0$ and

$$Ju(x) = zu(x), \quad x \neq x_0.$$

Then u is square summable on Γ .

Proof. Let $\mathcal{H}_0 = \mathbb{C}\delta_{x_0}$. The operator \tilde{J} acts on $\ell^2(\Gamma \setminus \{x_0\})$ and is not essentially selfadjoint by Lemma 6. Moreover if \tilde{u} denotes the truncation of u to $\tilde{\Gamma} = \Gamma \setminus \{x_0\}$ we have

$$\tilde{J}\tilde{u}(x) = z\tilde{u}(x), \quad x \in \tilde{\Gamma}.$$

By Lemma 1 we know that \tilde{u} cannot vanish. Since \tilde{J} is not essentially selfadjoint there exists a function $0 \neq \tilde{v} \in \ell^2(\tilde{\Gamma})$ so that

$$\tilde{J}\tilde{v}(x) = z\tilde{v}(x), \quad x \in \tilde{\Gamma}.$$

By Lemma 3 applied to $\tilde{\Gamma}$ we get that $\tilde{u}(x) = c\tilde{v}(x)$ for $x \in \tilde{\Gamma}$. \square

Fix an infinite path $\{x_n\}$ so that $l(x_n) = n$. By Corollary 2, for a nonreal number z there exist two nonzero solutions v_z and u_z on Γ such that

$$(25) \quad v_z(x_0) = 1, \quad v_z(x_1) = \frac{z - \beta_{x_0}}{\lambda_{x_0}}, \quad u_z(x_0) = 0, \quad u_z(x_1) = \frac{1}{\lambda_{x_0}}$$

$$(26) \quad Jv_z(x) = zv_z(x), \quad Ju_z(x) = zu_z(x), \quad x \in \Gamma \setminus \{x_0\}.$$

Observe that we have

$$Jv_z(x) = zv_z(x), \quad \text{for } x \in \Gamma.$$

The functions v_z and u_z satisfying (6) and (7) will be called *the solution* and *the associated solution* of the equation

$$Jf(x) = zf(x), \quad x \in \Gamma \setminus \{x_0\}.$$

Summarizing we get

Proposition 3. *Let J be nonessentially selfadjoint Jacobi matrix on Γ . Fix a vertex x_0 on the level 0. For any nonreal number z every solution of the equation*

$$zv(x) = \lambda_x v(x') + \beta_x v(x) + \sum_{y \in N_x} \lambda_y v(y), \quad x \neq x_0$$

is square summable.

Consider the graph obtained from Γ by removing all links (x_n, x_{n+1}) (we do not remove vertices). This graph splits into the infinite sum of finite subtrees Γ_n . The tree Γ_n contains the vertex x_n . Moreover x_n is the only vertex of Γ_n on the level n .

Lemma 7. *Let $x \in \Gamma_n$, for some $n \geq 1$. Then $v_z(x_n)u_z(x) = u_z(x_n)v_z(x)$.*

Proof. By Lemma 1 we know that v_z and u_z cannot vanish. Both functions satisfy $Ju_z(x) = zu_z(x)$, $Jv_z(x) = zv_z(x)$ for $x \in \Gamma_n \setminus \{x_n\}$. By Lemma 3 we get $v_z(x) = cu_z(x)$ for $x \in \Gamma_n$. Plugging in $x = x_n$ gives the conclusion. \square

Proposition 4. *For the solution v_z and the associated solution u_z we have*

$$\begin{vmatrix} v_z(x_n) & u_z(x_n) \\ v_z(x_{n+1}) & u_z(x_{n+1}) \end{vmatrix} = \frac{1}{\lambda_{x_n}}.$$

Proof. By the recurrence relation (6) we get

$$\begin{aligned} \lambda_{x_n} v_z(x_{n+1}) &= z v_z(x_n) - \beta_{x_n} v_z(x_n) - \lambda_{x_{n-1}} v_z(x_{n-1}) - \sum_{y \in N_{x_n} \setminus \{x_{n-1}\}} \lambda_y v_z(y), \\ \lambda_{x_n} u_z(x_{n+1}) &= z u_z(x_n) - \beta_{x_n} u_z(x_n) - \lambda_{x_{n-1}} u_z(x_{n-1}) - \sum_{y \in N_{x_n} \setminus \{x_{n-1}\}} \lambda_y u_z(y). \end{aligned}$$

On multiplying the equations by $u_z(x_n)$ and $v_z(x_n)$, respectively, subtracting sidewise and making use of Lemma 6 gives

$$\lambda_{x_n} \begin{vmatrix} v_z(x_n) & u_z(x_n) \\ v_z(x_{n+1}) & u_z(x_{n+1}) \end{vmatrix} = \lambda_{x_{n-1}} \begin{vmatrix} v_z(x_{n-1}) & u_z(x_{n-1}) \\ v_z(x_n) & u_z(x_n) \end{vmatrix}$$

The conclusion follows as

$$\lambda_{x_0} \begin{vmatrix} v_z(x_0) & u_z(x_0) \\ v_z(x_1) & u_z(x_1) \end{vmatrix} = 1.$$

□

Theorem 6. *Let J be a Jacobi matrix associated with the coefficients λ_x and β_x . Let x_n denote any infinite path so that $l(x_n) = n$. Assume*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_{x_n}} = \infty.$$

Then the operator J is essentially selfadjoint.

Proof. The result follows by the standard argument from Proposition 4. If J was not essentially selfadjoint then the functions v and u would be square summable, thus the series $\sum \lambda_{x_n}^{-1}$ would be summable. □

Remark. The assumption does not depend on the choice of the infinite path, as any two such paths will meet at a certain vertex.

5. NONNEGATIVE JACOBI MATRICES ON TREES

We say that a matrix J is positive definite if

$$(Jv, v) \geq 0, \quad v \in \mathcal{F}(\Gamma).$$

The next theorem gives characterization of positive definite Jacobi matrices on Γ .

Theorem 7. (i) Assume there exists a positive function $m(x)$ on Γ such that

$$(27) \quad \beta_x m(x) \geq \lambda_x m(x') + \sum_{y \in N_x} \lambda_y m(y), \quad x \in \Gamma.$$

Then the matrix J is positive definite

(ii) If the matrix J is positive definite there exists a positive function $m(x)$ on Γ such that

$$(28) \quad \beta_x m(x) = \lambda_x m(x') + \sum_{y \in N_x} \lambda_y m(y), \quad x \in \Gamma.$$

Proof. (i). For $x \in \Gamma$ let

$$\alpha_x = \lambda_x \frac{m(x)}{m(x')}, \quad \gamma_x = \lambda_x \frac{m(x')}{m(x)}.$$

Thus, on dividing by $m(x)$, the formula (27) takes the form

$$(29) \quad \beta_x \geq \gamma_x + \sum_{y \in N_x} \alpha_y, \quad x \in \Gamma.$$

We have (see (5))

$$\begin{aligned} (Jv, v) &= ((S + S^* + M)v, v) \\ &= \sum_{x \in \Gamma} \beta_x |v(x)|^2 + 2\operatorname{Re} \sum_{x \in \Gamma} \lambda_x \overline{v(x)} v(x') \\ &\geq \sum_{x \in \Gamma} \beta_x |v(x)|^2 - 2 \sum_{x \in \Gamma} \lambda_x |v(x)| |v(x')| \\ &= \sum_{x \in \Gamma} \beta_x |v(x)|^2 - 2 \sum_{x \in \Gamma} \sqrt{\alpha_x \gamma_x} |v(x)| |v(x')| \\ &\geq \sum_{x \in \Gamma} \beta_x |v(x)|^2 - \sum_{x \in \Gamma} \gamma_x |v(x)|^2 - \sum_{x \in \Gamma} \alpha_x |v(x')|^2 \\ &= \sum_{x \in \Gamma} \beta_x |v(x)|^2 - \sum_{x \in \Gamma} \gamma_x |v(x)|^2 - \sum_{x \in \Gamma} |v(x)|^2 \sum_{y \in N_x} \alpha_y \\ &= \sum_{x \in \Gamma} \left(\beta_x - \gamma_x - \sum_{y \in N_x} \alpha_y \right) |v(x)|^2 \geq 0. \end{aligned}$$

(ii) Consider the operator U acting by the rule

$$Uv(x) = (-1)^{l(x)} v(x).$$

Clearly U is a unitary operator. Let

$$\tilde{J} = -U^* J U.$$

Then \tilde{J} is a nonpositive definite operator and

$$\tilde{J}v(x) = \lambda_x v(x') - \beta_x v(x) + \sum_{y \in N_x} \lambda_y v(y).$$

Fix an infinite path x_n so that $l(x_n) = n$. Thus $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_{x_n}$. Let P_n denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_{x_n})$ and $\tilde{J}_n = P_n \tilde{J} P_n$. Then \tilde{J}_n is a bounded nonpositive linear operator. Therefore

$$-a_n I < \tilde{J}_n \leq 0 < \frac{1}{n} I,$$

for a positive constant a_n . Hence

$$0 < \tilde{J}_n + a_n I < (a_n + \frac{1}{n}) I.$$

We have

$$(30) \quad 0 < ((\tilde{J}_n + a_n I)\delta_x, \delta_x) = a_n - \beta_x, \quad x \in \Gamma_{x_n}.$$

Observe that

$$(31) \quad (\tilde{J}_n + a_n I)\delta_x = \lambda_x \delta_{x'} + (a_n - \beta_x)\delta_x + \sum_{y \in N_x} \lambda_y \delta_y, \quad x \in \Gamma_{x_n} \setminus \{x_n\}.$$

Let

$$\begin{aligned} f_n &:= (\frac{1}{n} I - \tilde{J}_n)^{-1} \delta_{x_0} = [(a_n + \frac{1}{n}) I - (\tilde{J}_n + a_n I)]^{-1} \delta_{x_0} \\ &= \sum_{k=0}^{\infty} \frac{1}{(a_n + \frac{1}{n})^{k+1}} (\tilde{J}_n + a_n I)^k \delta_{x_0}. \end{aligned}$$

By (30) and (31) the function

$$(\tilde{J}_n + a_n I)^k \delta_{x_0}$$

is nonnegative, and positive on all vertices of Γ_{x_n} at distance from x_0 less or equal to k . Hence $f_n \geq 0$ and $f_n(x) > 0$ for any $x \in \Gamma_{x_n}$. Moreover

$$\tilde{J}_n f_n = \tilde{J}_n (\frac{1}{n} I - \tilde{J}_n)^{-1} \delta_{x_0} = (\tilde{J}_n - \frac{1}{n} I) (\frac{1}{n} I - \tilde{J}_n)^{-1} \delta_{x_0} + \frac{1}{n} f_n = \frac{1}{n} f_n - \delta_{x_0}.$$

This results in

$$\lambda_x f_n(x') - \beta_x f_n(x) + \sum_{y \in N_x} \lambda_y f_n(y) = \frac{1}{n} f_n(x) - \delta_{x_0}(x), \quad x \in \Gamma_{x_n} \setminus \{x_n\}.$$

Let

$$m_n(x) = \frac{f_n(x)}{f_n(x_0)}.$$

Then $m_n(x_0) = 1$ and

$$(32) \quad \lambda_x m_n(x') + \sum_{y \in N_x} \lambda_y m_n(y) = (\beta_x + \frac{1}{n}) m_n(x), \quad x \in \Gamma_{x_n} \setminus \{x_0, x_n\},$$

$$(33) \quad \lambda_{x_0} m_n(x_1) \leq (\beta_{x_0} + \frac{1}{n}).$$

Observe that for any fixed $t \in \Gamma$ the sequence $m_n(t)$ is bounded. Indeed, assume the opposite. Let t be the vertex closest to x_0 , so that $m_n(t)$ is unbounded. Let s be the vertex adjacent to t , so that

$$d(x_0, t) = d(x_0, s) + 1.$$

Then applying (32) with $x = s$ implies that the sequence $m_n(s)$ is unbounded, which gives a contradiction.

Observe also that for any fixed $t \in \Gamma$ the sequence $m_n(t)$ cannot accumulate at zero. Indeed, assume the opposite. Let t be the vertex closest to x_0 so that $m_n(t)$ accumulates at zero. Again let s be the vertex adjacent to t , so that

$$d(x_0, t) = d(x_0, s) + 1.$$

Then applying (32) with $x = t$ implies that the sequence $m_n(s)$ also accumulates at zero, which gives a contradiction.

Consider the sequence of functions m_n . Let m be any pointwise accumulation point of this sequence. Then $m(x) > 0$ and by (32) and (33) we obtain

$$(34) \quad \lambda_x m(x') + \sum_{y \in N_x} \lambda_y m(y) = \beta_x m(x), \quad x \in \Gamma \setminus \{x_0\},$$

$$(35) \quad \lambda_{x_0} m(x_1) \leq \beta_{x_0}.$$

In order to get the conclusion (i.e. to guarantee equality also in (35)) we have to modify slightly the function $m(x)$.

Observe that after removing all the edges from the path $\{x_n\}$ the tree Γ splits into the sequence of disjoint trees Γ_n so that $x_n \in \Gamma_n$. By (34) evaluated at $x = x_n$ we have

$$\lambda_{x_n} m(x_{n+1}) + \lambda_{x_{n-1}} m(x_{n-1}) + \sum_{\substack{y \in N_{x_n} \\ y \neq x_{n-1}}} \lambda_y m(y) = \beta_{x_n} m(x_n), \quad n \geq 1.$$

Let the coefficients c_n be defined by $c_0 = 0$ and

$$(36) \quad \sum_{\substack{y \in N_{x_n} \\ y \neq x_{n-1}}} \lambda_y m(y) = c_n m(x_n), \quad n \geq 1.$$

Thus

$$\lambda_{x_n} m(x_{n+1}) + \lambda_{x_{n-1}} m(x_{n-1}) = (\beta_{x_n} - c_n) m(x_n), \quad n \geq 1.$$

This implies $\beta_{x_n} \geq c_n$. Consider the classical Jacobi matrix defined by

$$J_0 u(n) = \lambda_{x_n} u(n+1) + (\beta_{x_n} - c_n) u(n) + \lambda_{x_{n-1}} u(n-1).$$

By Theorem 9(i) the matrix J_0 is positive definite. Let p_n denote the orthogonal polynomials associated with J_0 . By the well known result we have $(-1)^n p_n(0) > 0$. Set $\tilde{m}(x_n) = (-1)^n p_n(0)$. Then

$$(37) \quad \lambda_{x_n} \tilde{m}(x_{n+1}) + \lambda_{x_{n-1}} \tilde{m}(x_{n-1}) = (\beta_{x_n} - c_n) \tilde{m}(x_n).$$

Set also

$$(38) \quad \tilde{m}(y) = \frac{\tilde{m}(x_n)}{m(x_n)} m(y), \quad y \in \Gamma_n.$$

In view of (36), (37) and (38) we get

$$\lambda_{x_n} \tilde{m}(x_{n+1}) + \sum_{y \in N_{x_n}} \lambda_y \tilde{m}(y) = \beta_{x_n} \tilde{m}(x_n), \quad n \geq 0.$$

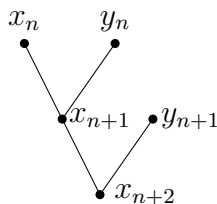
Finally, by (34) and (38) we have

$$\lambda_x m(x') + \sum_{y \in N_x} \lambda_y m(y) = \beta_x m(x), \quad x \in \Gamma_n \setminus \{x_n\}, n \geq 1.$$

□

Proposition 5. *There exist Jacobi matrices J on trees so that the equation $Jv = tv$ does not admit nonzero solutions for some real values of t .*

Proof. We may admit that $t = 0$. Consider a tree Γ with $\#N_x = 2$ for every vertex x , $\ell(x) \geq 1$. Fix an infinite path x_n , so that $\ell(x_n) = n$. Then $N_{x_n} = \{x_{n-1}, y_{n-1}\}$ for $n \geq 1$. We will define the coefficients λ_x and β_x on Γ_{y_k} in such a way that the operator J restricted to $\ell^2(\Gamma_{y_k} \setminus \{y_k\})$ is positive. For example we may set $\lambda_x = 1$ and $\beta_x = 4$ for any $x \in \Gamma_{y_k} \setminus \{y_k\}$. In this way if the function v satisfies $Jv(y) = 0$ for $y \in \Gamma_{y_k} \setminus \{y_k\}$, then either $v = 0$ or v cannot vanish on Γ_{y_k} . If v does not vanish on Γ_{y_k} its restriction to Γ_{y_k} is unique up to a constant multiple. Let $\lambda_{y_k} = 1$ and set β_{y_k} in such a way that $v(y'_k) = v(x_{k+1}) = 0$. Set also $\lambda_{x_k} = 1$ and $\beta_{x_k} = 0$ for any k . Thus the matrix J is defined. Assume $Ju = 0$. If u vanishes on every subtree Γ_{y_n} then by the recurrence relation u vanishes at every vertex x_n , with $n \geq 1$, as $y'_n = x_{n+1}$. Moreover by the recurrence relation evaluated at x_1 we obtain $v(x_0) = 0$, i.e. $v = 0$. If u does not vanish on every subtree Γ_{y_n} , let n be the smallest index for which u does not vanish on Γ_{y_n} .



Then $v(x_k) = 0$ for any $k \leq n$. We must have $v(y_n) \neq 0$. By construction we also get $v(x_{n+1}) = 0$. By the recurrence relation evaluated at x_{n+1} we conclude that $v(x_{n+2}) \neq 0$. This implies that v does not vanish on $\Gamma_{y_{n+1}}$. But by construction $v(x_{n+2}) = 0$, which is a contradiction. \square

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